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Generalized Nonlinear Mixed Quasi-Variational Inequalities

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Abstract—In this paper, we introduce and study a new class of quasi-variational inequalities, which is called the generalized nonlinear set-valued mixed quasi-variational inequality. Using the resolvent operator technique for maximal monotone mapping, we construct some new iterative algorithms for solving this class of generalized nonlinear set-valued mixed quasi-variational inequalities. We prove the existence of solution for this kind of generalized nonlinear set-valued mixed quasi-variational inequalities without compactness and the convergence of iterative sequences generated by the algorithms. We also discuss the convergence and stability of perturbed iterative algorithm for solving a class of generalized nonlinear mixed quasi-variational inequalities.

Keywords—Mixed quasi-variational inequality, Set-valued mapping, Iterative algorithm, Perturbed algorithm, Stability.

1. INTRODUCTION

Variational inequality theory and complementarity problem theory are very powerful tools of the current mathematical technology. In recent years, classical variational inequality and complementarity problems have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, regional, structural, transportation, elasticity, and applied sciences, etc. (See [1–42] and the references therein.) A useful and an important generation of variational inequalities is a mixed variational inequality containing a nonlinear term. Due to the presence of the nonlinear term, the projection method cannot be used to study the existence of a solution for the mixed variational inequalities. In 1994, Hassouni and Moudafi [17] used the resolvent operator technique for maximal monotone mapping to study a new class of mixed variational inequalities for

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single-valued mappings. In 1996, Huang [20] extended this technique for a new class of general mixed variational inequalities with noncompact set-valued mappings and Adly [1] modified this technique for another new class of general mixed variational inequalities for single-valued mappings, which includes the mixed variational inequality considered by Hassouni and Moudafi [17] as special cases. For related works, we refer to [12,19,22,27,33,39].

Inspired and motivated by recent research works [1,18–20,22,33,39], in this paper, we introduce and study a new class of quasi-variational inequalities, which is called the generalized nonlinear set-valued mixed quasi-variational inequality. We establish the equivalence between generalized nonlinear set-valued mixed quasi-variational inequality and fixed-point problems by employing the resolvent operator technique for maximal monotone mapping. Using this equivalence, we construct some new iterative algorithms for solving this class of generalized nonlinear set-valued mixed quasi-variational inequalities. We prove some existence theorems of solution for this kind of generalized nonlinear set-valued mixed quasi-variational inequalities without compactness and the convergence of iterative sequences generated by the algorithms. We also discuss the convergence and stability of perturbed iterative algorithm for solving a class of generalized nonlinear mixed quasi-variational inequalities. The results shown in this paper improve and extend the previously known results in this area.

2. PRELIMINARIES

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $G, S, T : H \to 2^H$ be set-valued mappings, where 2^H denotes the family of all nonempty subsets of H, and $p : H \to H$ and $N : H \times H \to H$ be single-valued mappings. Suppose that $M : H \times H \to 2^H$ is a set-valued mapping such that for each fixed $t \in H$, $M(\cdot, t) : H \to 2^H$ is a maximal monotone mapping and $\mathrm{Range}(p) \cap \mathrm{dom}(M(\cdot,t)) \neq \emptyset$ for each $t \in H$. We consider the following problem.

Find $u \in H$, $x \in Su$, $y \in Tu$, $z \in Gu$ such that $p(u) \in \mathrm{dom}(M(\cdot, z))$ and

$$0 \in N(x,y) + M(p(u), z). \tag{2.1}$$

Problem (2.1) is called a *generalized nonlinear set-valued mixed quasi-variational inequality*.

A well-known example [43] of a maximal monotone mapping is the subdifferential of a proper lower semicontinuous convex function. Letting $M(\cdot,t) = \partial\varphi(\cdot,t)$ for each $t \in H$, where $\varphi : H \times H \to R \cup \{+\infty\}$ such that for each fixed $t \in H$, $\varphi(\cdot,t) : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $p(H) \cap \mathrm{dom}(\partial\varphi(\cdot,t)) \neq \emptyset$ for each $t \in H$ and $\partial\varphi(\cdot,t)$ denotes the subdifferential of function $\varphi(\cdot,t)$, then problem (2.1) is equivalent to finding $u \in H$, $x \in Su$, $y \in Tu$, $z \in Gu$ such that

$$\begin{gathered} p(u) \in \mathrm{dom}\left(\partial\varphi\left(\cdot, z\right)\right), \\ \langle N(x,y), v - p(u)\rangle \geq \varphi(p(u), z) - \varphi(v, z), \end{gathered} \tag{2.2}$$

for all $v \in H$.

If G is an identity mapping and $M(\cdot,t) = \partial\varphi$ for each $t \in H$, where $\varphi : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $p(H) \cap \mathrm{dom}(\partial\varphi)) \neq \emptyset$ and $\partial\varphi$ denotes the subdifferential of function φ, then problem (2.1) is equivalent to finding $u \in H$, $x \in Su$, $y \in Tu$ such that

$$\begin{gathered} p(u) \in \mathrm{dom}\left(\partial\varphi\right), \\ \langle N(x,y), v - p(u)\rangle \geq \varphi(p(u)) - \varphi(v), \end{gathered} \tag{2.3}$$

for all $v \in H$. Problem (2.3) is called the *generalized set-valued mixed variational inequality*, which was studied by Noor, Noor and Rassias [33]. It is known that a number of problems involving the nonmonotone, nonconvex, and nonsmooth mappings arising in structural engineering, mechanics, economics, and optimization theory can be studied via problem (2.3); see, for example, [11,15] and the references therein.

It is well known [6,44] that there exist maximal monotone mappings which are not subdifferentials of lower semicontinuous proper convex functions. Therefore, problem (2.1) is more general than problems (2.2) and (2.3).

If G is the identity mapping and $M(s,t) = M(s)$ for all $t \in H$, where $M : H \to 2^H$ is a maximal monotone mapping, then problem (2.1) is equivalent to finding $u \in H$, $x \in Su$, $y \in Tu$ such that $p(u) \in \operatorname{dom}(M)$ and

$$0 \in N(x,y) + M(p(u)). \tag{2.4}$$

Problem (2.4) is called the *generalized set-valued mixed variational inequality*, which was studied by Huang [22].

If S and T are single-valued mappings and G is an identity mapping, then problem (2.1) is equivalent to finding $u \in H$ such that $p(u) \in \operatorname{dom}(M(\cdot,u))$ and

$$0 \in N(Su,Tu) + M(p(u),u), \tag{2.5}$$

which is called the *generalized nonlinear mixed quasi-variational inequality*.

If S and T are single-valued mappings, G is an identity mapping, $N(x,y) = x - y$, $p = w - m$, $M(s,t) = M(s)$ for all $t \in H$, where w and m are single-valued mappings, and $M : H \to 2^H$ is a maximal monotone mapping, then problem (2.1) is equivalent to finding $u \in H$ such that $w(u) - m(u) \in \operatorname{dom}(M))$ and

$$0 \in Su - Tu + M(w(u) - m(u)). \tag{2.6}$$

This problem is called the *generalized mixed variational inequality*, which was studied by Uko [39].

For a suitable choice of the mappings S, T, G, N, p, M, and the space H, a number of known classes mixed variational inequalities, variational inequalities, quasi-variational inequalities, complementarity problems, and quasi-(implicit) complementarity problems of [1,6–8,12,13,17–25,27,30–34,36–39,41] can be obtained as special cases of the generalized nonlinear set-valued mixed quasi-variational inequality (2.1). Further, these type of quasi-variational inequalities enable us to study many important problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, regional, structural, transportation, elasticity, and applied sciences in a general and unified framework.

3. ITERATIVE ALGORITHMS

It is well known (cf. [6,43]) that if M is a maximal monotone mapping from H to 2^H, then for every $\mu > 0$, the resolvent $(I + \mu M)^{-1}$ is a well-defined single-valued nonexpansive operator mapping H into itself. By using the resolvent operator technique, it is possible to convert the generalized nonlinear set-valued mixed quasi-variational inequality (2.1) into an equivalent equation which is easier to handle. To do this, we multiply all the terms in (2.1) with some $\rho > 0$ and add $p(u)$. We obtain

$$p(u) - \rho N(x,y) \in p(u) + \rho M(p(u),z).$$

Therefore, we have the following.

LEMMA 3.1. *(u, x, y, z) is a solution of problem (2.1) if and only if (u,x,y,z) satisfies the relation*

$$p(u) = J_\rho^{M(\cdot,z)}(p(u) - \rho N(x,y)),$$

where $\rho > 0$ is a constant, $J_\rho^{M(\cdot,z)} = (I + \rho M(\cdot,z))^{-1}$, and I is an identity mapping on H.

Based on Lemma 3.1 and Nadler's result [45], we now suggest and analyze the following new general and unified algorithms for problem (2.1).

Let $p : H \to H$, $N : H \times H \to H$, and $S, T, G : H \to CB(H)$, where $CB(H)$ is the family of all nonempty bounded closed subsets of H. For given $u_0 \in H$, we take $x_0 \in Su_0$, $y_0 \in Tu_0$, and $z_0 \in Gu_0$, and let

$$u_1 = u_0 - p(u_0) + J_\rho^{M(\cdot, z_0)} (p(u_0) - \rho N(x_0, y_0)).$$

Since $x_0 \in Su_0 \in CB(H)$, $y_0 \in Tu_0 \in CB(H)$, and $z_0 \in Gu_0 \in CB(H)$, by Nadler [45], there exist $x_1 \in Su_1$, $y_1 \in Tu_1$, and $z_1 \in Gu_1$ such that

$$\begin{aligned} \|x_0 - x_1\| &\le (1+1) H(Su_0, Su_1), \\ \|y_0 - y_1\| &\le (1+1) H(Tu_0, Tu_1), \\ \|z_0 - z_1\| &\le (1+1) H(Gu_0, Gu_1), \end{aligned}$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$. By induction, we can obtain our algorithm for problem (2.1) as follows.

ALGORITHM 3.1. *Suppose that $p : H \to H$, $N : H \times H \to H$, and $S, T, G : H \to CB(H)$. For given $u_0 \in H$, $x_0 \in Su_0$, $y_0 \in Tu_0$, and $z_0 \in Gu_0$, compute $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ from the iterative schemes*

$$\begin{aligned} u_{n+1} &= u_n - p(u_n) + J_\rho^{M(\cdot, z_n)} (p(u_n) - \rho N(x_n, y_n)), \\ \|x_n - x_{n+1}\| &\le \left(1 + (n+1)^{-1}\right) H(Su_n, Su_{n+1}), \qquad x_n \in Su_n, \\ \|y_n - y_{n+1}\| &\le \left(1 + (n+1)^{-1}\right) H(Tu_n, Tu_{n+1}), \qquad y_n \in Tu_n, \\ \|z_n - z_{n+1}\| &\le \left(1 + (n+1)^{-1}\right) H(Gu_n, Gu_{n+1}), \qquad z_n \in Gu_n, \end{aligned} \tag{3.1}$$

for $n = 0, 1, 2, \ldots$, where $\rho > 0$ is a constant.

From Algorithm 3.1, we can get an algorithm for problem (2.2) as follows.

ALGORITHM 3.2. Suppose that $p : H \to H$, $N : H \times H \to H$, and $S, T, G : H \to CB(H)$. For given $u_0 \in H$, $x_0 \in Su_0$, $y_0 \in Tu_0$, and $z_0 \in Gu_0$, compute $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ from the iterative schemes

$$\begin{aligned} u_{n+1} &= u_n - p(u_n) + J_\rho^{\partial\varphi(\cdot, z_n)} (p(u_n) - \rho N(x_n, y_n)), \\ \|x_n - x_{n+1}\| &\le \left(1 + (n+1)^{-1}\right) H(Su_n, Su_{n+1}), \qquad x_n \in Su_n, \\ \|y_n - y_{n+1}\| &\le \left(1 + (n+1)^{-1}\right) H(Tu_n, Tu_{n+1}), \qquad y_n \in Tu_n, \\ \|z_n - z_{n+1}\| &\le \left(1 + (n+1)^{-1}\right) H(Gu_n, Gu_{n+1}), \qquad z_n \in Gu_n, \end{aligned} \tag{3.2}$$

for $n = 0, 1, 2, \ldots$, where $\rho > 0$ is a constant and $J_\rho^{\partial\varphi(\cdot, z)} = (I + \rho\partial\varphi(\cdot, z))^{-1}$.

For a suitable choice of the mappings S, T, G, N, p, M, and the space H, many known iterative algorithms for solving various classes of variational inequalities and complementarity problems in [1,6–8,12,13,17–25,27,30–34,36,37–39,41] can be obtained as special cases of Algorithms 3.1 and 3.2.

4. EXISTENCE AND CONVERGENCE THEOREMS

In this section, we prove the existence of a solution of problem (2.1) and the convergence of an iterative sequence generated by Algorithm 3.1.

DEFINITION 4.1. *A mapping $g : H \to H$ is said to be*

(1) *strongly monotone if there exists some $\delta > 0$ such that*

$$\langle g(u_1) - g(u_2), u_1 - u_2 \rangle \ge \delta \|u_1 - u_2\|^2,$$

for all $u_i \in H$, $i = 1, 2$;

(2) *Lipschitz continuous if there exists some $\sigma > 0$ such that*

$$\|g(u_1) - g(u_2)\| \le \sigma \|u_1 - u_2\|,$$

for all $u_i \in H$, $i = 1, 2$.

DEFINITION 4.2. *A set-valued mapping $S : H \to CB(H)$ is said to be as follows.*

(1) *H-Lipschitz continuous if there exists a constant $\eta > 0$ such that*

$$H(S(u_1), S(u_2)) \le \eta \|u_1 - u_2\|,$$

for all $u_i \in H$, $i = 1, 2$;

(2) *strongly monotone with respect to the first argument of $N(\cdot,\cdot) : H \times H \to H$, if there exists a constant $\alpha > 0$ such that*

$$\langle N(x_1,\cdot) - N(x_2,\cdot), u_1 - u_2\rangle \ge \alpha \|u_1 - u_2\|^2$$

for all $x_i \in S(u_i)$, $i = 1, 2$.

DEFINITION 4.3. *The operator $N : H \times H \to H$ is said to be Lipschitz continuous with respect to the first argument if there exists a constant $\beta > 0$ such that*

$$\|N(u_1,\cdot) - N(u_2,\cdot)\| \le \beta \|u_1 - u_2\|,$$

for all $u_i \in H$, $i = 1, 2$.

In a similar way, we can define Lipschitz continuity of the operator $N(\cdot,\cdot)$ with respect to the second argument.

THEOREM 4.1. *Let N be Lipschitz continuous with respect to the first and second arguments with constants β, ξ, respectively. Let $S : H \to CB(H)$ be strongly monotone with respect to the first argument of $N(\cdot,\cdot)$ with constant α. Let $S, T, G : H \to CB(H)$ be H-Lipschitz with constants η, γ, and s, respectively, $p : H \to H$ be strongly monotone and Lipschitz continuous with constants δ and σ, respectively. Suppose that there exist constants $\lambda > 0$ and $\rho > 0$ such that, for each x, y, $z \in H$,*

$$\left\| J_\rho^{M(\cdot,x)}(z) - J_\rho^{M(\cdot,y)}(z) \right\| \le \lambda \|x - y\| \tag{4.1}$$

and

$$\begin{gathered} \left| \rho - \frac{\alpha + \xi\gamma(k-1)}{\eta^2\beta^2 - \xi^2\gamma^2} \right| < \frac{\sqrt{(\alpha + \xi\gamma(k-1))^2 - (\eta^2\beta^2 - \xi^2\gamma^2)\, k(2-k)}}{\eta^2\beta^2 - \xi^2\gamma^2}, \\ \alpha > (1-k)\xi\gamma + \sqrt{(\eta^2\beta^2 - \xi^2\gamma^2)\, k(2-k)}, \qquad \eta\beta > \xi\gamma, \\ \rho\xi\gamma < 1 - k, \quad k = \lambda s + 2\sqrt{1 - 2\delta + \sigma^2}, \qquad k < 1. \end{gathered} \tag{4.2}$$

Then there exist $u \in H$, $x \in Su$, $y \in Tu$, and $z \in Gu$ satisfying problem (2.1). Moreover,

$$u_n \to u, \quad x_n \to x, \quad y_n \to y, \quad z_n \to z, \qquad \text{as } n \to \infty,$$

where $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are sequences defined in Algorithm 3.1.

PROOF. From Algorithm 3.1 and (4.1), we have

$$\begin{aligned} \|u_{n+1} - u_n\| &= \Big\| u_n - u_{n-1} - (p(u_n) - p(u_{n-1})) + J_\rho^{M(\cdot,z_n)}(p(u_n) - \rho N(x_n, y_n)) \\ &\qquad - J_\rho^{M(\cdot,z_{n-1})}(p(u_{n-1}) - \rho N(x_{n-1}, y_{n-1})) \Big\| \\ &\le \|u_n - u_{n-1} - (p(u_n) - p(u_{n-1}))\| \\ &\qquad + \left\| J_\rho^{M(\cdot,z_n)}(p(u_n) - \rho N(x_n, y_n)) - J_\rho^{M(\cdot,z_{n-1})}(p(u_{n-1}) - \rho N(x_{n-1}, y_{n-1})) \right\| \\ &\le \|u_n - u_{n-1} - (p(u_n) - p(u_{n-1}))\| \end{aligned} \tag{4.3}$$

$$
\begin{aligned}
&+\left\|J_\rho^{M(\cdot,z_n)}\left(p\left(u_{n-1}\right)-\rho N\left(x_{n-1},y_{n-1}\right)\right)-J_\rho^{M(\cdot,z_{n-1})}\left(p\left(u_{n-1}\right)\right.\right.\\
&\left.\left.-\rho N\left(x_{n-1},y_{n-1}\right)\right)\right\|\\
&+\left\|J_\rho^{M(\cdot,z_n)}\left(p\left(u_n\right)-\rho N\left(x_n,y_n\right)\right)-J_\rho^{M(\cdot,z_n)}\left(p\left(u_{n-1}\right)-\rho N\left(x_{n-1},y_{n-1}\right)\right)\right\|\\
\leq{}&\left\|u_n-u_{n-1}-\left(p\left(u_n\right)-p\left(u_{n-1}\right)\right)\right\|+\lambda\left\|z_n-z_{n-1}\right\|\\
&+\left\|\left(p\left(u_n\right)-\rho N\left(x_n,y_n\right)\right)-\left(p\left(u_{n-1}\right)-\rho N\left(x_{n-1},y_{n-1}\right)\right)\right\|\\
\leq{}&2\left\|u_n-u_{n-1}-\left(p\left(u_n\right)-p\left(u_{n-1}\right)\right)\right\|+\lambda\left\|z_n-z_{n-1}\right\|\\
&+\left\|u_n-u_{n-1}-\rho\left(N\left(x_n,y_n\right)-N\left(x_{n-1},y_{n-1}\right)\right)\right\|\\
\leq{}&2\left\|u_n-u_{n-1}-\left(p\left(u_n\right)-p\left(u_{n-1}\right)\right)\right\|+\lambda\left\|z_n-z_{n-1}\right\|\\
&+\left\|u_n-u_{n-1}-\rho\left(N\left(x_n,y_n\right)-N\left(x_{n-1},y_n\right)\right)\right\|\\
&+\rho\left\|N\left(x_{n-1},y_n\right)-N\left(x_{n-1},y_{n-1}\right)\right\|.
\end{aligned}
\tag{4.3)(cont.}
$$

By the Lipschitz continuity and strong monotonicity of p, we obtain

$$
\left\|u_n-u_{n-1}-\left(p\left(u_n\right)-p\left(u_{n-1}\right)\right)\right\|^2 \leq\left(1-2\delta+\sigma^2\right)\left\|u_n-u_{n-1}\right\|^2. \tag{4.4}
$$

Since S is H-Lipschitz continuous and strongly monotone with respect to the first argument of N and N is Lipschitz continuous with respect to the first argument, we have

$$
\begin{aligned}
&\left\|u_n-u_{n-1}-\rho\left(N\left(x_n,y_n\right)-N\left(x_{n-1},y_n\right)\right)\right\|^2\\
={}&\left\|u_n-u_{n-1}\right\|^2-2\rho\left\langle u_n-u_{n-1},N\left(x_n,y_n\right)-N\left(x_{n-1},y_n\right)\right\rangle\\
&+\rho^2\left\|N\left(x_n,y_n\right)-N\left(x_{n-1},y_n\right)\right\|^2\\
\leq{}&\left(1-2\rho\alpha+\rho^2\eta^2\left(1+n^{-1}\right)^2\beta^2\right)\left\|u_n-u_{n-1}\right\|^2.
\end{aligned}
\tag{4.5}
$$

Further, since T, G are H-Lipschitz continuous and N is Lipschitz continuous with respect to the second argument, we get

$$
\left\|N\left(x_{n-1},y_n\right)-N\left(x_{n-1},y_{n-1}\right)\right\| \leq \xi\left\|y_n-y_{n-1}\right\| \leq \xi\gamma\left(1+n^{-1}\right)\left\|u_n-u_{n-1}\right\|, \tag{4.6}
$$

$$
\left\|z_n-z_{n-1}\right\| \leq s\left(1+n^{-1}\right)\left\|u_n-u_{n-1}\right\|. \tag{4.7}
$$

From (4.3)–(4.7), it follows that

$$
\left\|u_n-u_{n+1}\right\| \leq \theta_n\left\|u_n-u_{n-1}\right\|, \tag{4.8}
$$

where

$$
\theta_n=\lambda s\left(1+n^{-1}\right)+2\sqrt{1-2\delta+\sigma^2}+\sqrt{1-2\rho\alpha+\rho^2\eta^2\beta^2\left(1+n^{-1}\right)^2}+\rho\xi\gamma\left(1+n^{-1}\right).
$$

Letting

$$
\theta=k+\sqrt{1-2\rho\alpha+\rho^2\eta^2\beta^2}+\rho\xi\gamma,
$$

where $k=\lambda s+2\sqrt{1-2\delta+\sigma^2}$, we know $\theta_n \searrow \theta$. It follows from (4.2) that $\theta<1$. Hence, $\theta_n<1$, for n sufficiently large. Therefore, (4.8) implies that $\{u_n\}$ is a Cauchy sequence in H and we can suppose that $u_n \to u \in H$.

Now we prove that $x_n \to x \in Su$, $y_n \to y \in Tu$, and $z_n \to z \in Gu$. In fact, it follows from Algorithm 3.1 that

$$
\begin{aligned}
\left\|x_n-x_{n-1}\right\| &\leq\left(1+n^{-1}\right)\eta\left\|u_n-u_{n-1}\right\|,\\
\left\|y_n-y_{n-1}\right\| &\leq\left(1+n^{-1}\right)\gamma\left\|u_n-u_{n-1}\right\|,\\
\left\|z_n-z_{n-1}\right\| &\leq\left(1+n^{-1}\right)\sigma\left\|u_n-u_{n-1}\right\|,
\end{aligned}
$$

which imply that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are all Cauchy sequences. Let $x_n \to x$, $y_n \to y$, and $z_n \to z$. Further, we have

$$\begin{aligned} d(x, Su) &= \inf\{\|x - v\| : v \in Su\} \le \|x - x_n\| + d(x_n, Su) \\ &\le \|x - x_n\| + H(Su_n, Su) \le \|x - x_n\| + \eta \|u_n - u\| \to 0. \end{aligned}$$

Hence, we have $x \in Su$. Similarly, we have $y \in Tu$ and $z \in Gu$. This completes the proof.

From Theorem 4.1, we have the following result.

THEOREM 4.2. *Let N, S, T, G, p be the same as in Theorem 4.1. Suppose that there exist constants $\lambda > 0$ and $\rho > 0$ such that, for each x, y, $z \in H$,*

$$\left\| J_\rho^{\partial\varphi(\cdot,x)}(z) - J_\rho^{\partial\varphi(\cdot,y)}(z) \right\| \le \lambda \|x - y\|$$

and condition (4.2) in Theorem 4.1 holds. Then there exist $u \in H$, $x \in Su$, $y \in Tu$, and $z \in Gu$, satisfying problem (2.2). Moreover,

$$u_n \to u, \quad x_n \to x, \quad y_n \to y, \quad z_n \to z, \qquad \text{as } n \to \infty,$$

where $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are sequences defined in Algorithm 3.2.

For an appropriate and suitable choice of the mappings S, T, G, N, p, M, and the space H, we can obtain several known results in [1,6–8,12,13,17–25,27,30–34,36,37–39,41] as special cases of Theorems 4.1 and 4.2.

5. PERTURBED ALGORITHMS AND STABILITY

In this section, we construct a new perturbed iterative algorithm for solving the generalized nonlinear mixed quasi-variational inequality (2.5) and prove the convergence and stability of the iterative sequence generated by the perturbed iterative algorithm.

DEFINITION 5.1. *Let T be a selfmap of H, $x_0 \in H$ and let $x_{n+1} = f(T, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^\infty$ in H. Suppose that $\{x \in H : Tx = x\} \neq \emptyset$ and $\{x_n\}_{n=0}^\infty$ converges to a fixed point x^* of T. Let $\{y_n\} \subset H$ and let $\epsilon_n = \|y_{n+1} - f(T, y_n)\|$. If $\lim \epsilon_n = 0$ implies that $\lim y_n = x^*$, then the iteration procedure defined by $x_{n+1} = f(T, x_n)$ is said to be T-stable or stable with respect to T.*

Harder and Hicks [46] showed how such a sequence $\{y_n\}$ could arise in practice and demonstrated the importance of investigating the stability of various iterative procedures for various classes of nonlinear mappings. For details, the reader may consult [47] and the references therein.

For our results, we need the following lemma.

LEMMA 5.1. *(See [48].) Let $\{\gamma_n\}$ be a nonnegative real sequence and $\{\lambda_n\}$ be a real sequence in $[0, 1]$ such that $\sum_{n=0}^\infty \lambda_n = \infty$. If there exists a positive integer n_1 such that*

$$\gamma_{n+1} \le (1 - \lambda_n) \gamma_n + \lambda_n \sigma_n,$$

for all $n \ge n_1$, where $\sigma_n \ge 0$ for all $n \ge 0$ and $\sigma_n \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} \gamma_n = 0$.

ALGORITHM 5.1. Let S, T, $p : H \to H$ be single-valued mappings. For given $u_0 \in H$, the perturbed iterative sequence $\{u_n\}$ is defined by

$$\begin{aligned} u_{n+1} &= (1 - \alpha_n) u_n + \alpha_n \left[v_n - p(v_n) + J_\rho^{M(\cdot, v_n)} (p(v_n) - \rho N(Sv_n, Tv_n)) \right] + \alpha_n e_n, \\ v_n &= (1 - \beta_n) u_n + \beta_n \left[u_n - p(u_n) + J_\rho^{M(\cdot, u_n)} (p(u_n) - \rho N(Su_n, Tu_n)) \right] + \beta_n f_n, \end{aligned} \tag{5.1}$$

for $n = 0, 1, 2, \ldots$, where $\{e_n\}$ and $\{f_n\}$ are two sequences of the elements of H introduced to take into account possible inexact computation and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

$$0 \leq \alpha_n, \quad \beta_n \leq 1, \quad (n \geq 0), \quad \text{and} \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Let $\{y_n\}$ be any sequence in H and define $\{\epsilon_n\}$ by

$$\begin{aligned} \epsilon_n &= \left\| y_{n+1} - \left\{ (1-\alpha_n) y_n + \alpha_n \left[x_n - p(x_n) + J_\rho^{M(\cdot, x_n)} \left(p(x_n) - \rho N(Sx_n, Tx_n) \right) \right] + \alpha_n e_n \right\} \right\|, \\ x_n &= (1 - \beta_n) y_n + \beta_n \left[y_n - p(y_n) + J_\rho^{M(\cdot, y_n)} \left(p(y_n) - \rho N(Sy_n, Ty_n) \right) \right] + \beta_n f_n, \end{aligned} \tag{5.2}$$

for $n = 0, 1, 2, \ldots$.

THEOREM 5.1. *Let N be Lipschitz continuous with respect to the first and second arguments with constants β, ξ, respectively. Let $S : H \to H$ be strongly monotone with respect to the first argument of N with constant α. Let $S, T : H \to H$ be Lipschitz continuous with constants η and γ, respectively, $p : H \to H$ be strongly monotone and Lipschitz continuous with constants δ and σ, respectively. Suppose that there exist constants $\lambda > 0$ and $\rho > 0$ such that, for each x, y, $z \in H$,*

$$\left\| J_\rho^{M(\cdot, x)}(z) - J_\rho^{M(\cdot, y)}(z) \right\| \leq \lambda \|x - y\| \tag{5.3}$$

and

$$\begin{aligned} &\left| \rho - \frac{\alpha + \xi\gamma(k-1)}{\eta^2\beta^2 - \xi^2\gamma^2} \right| < \frac{\sqrt{(\alpha + \xi\gamma(k-1))^2 - (\eta^2\beta^2 - \xi^2\gamma^2) k(2-k)}}{\eta^2\beta^2 - \xi^2\gamma^2}, \\ &\alpha > (1-k)\xi\gamma + \sqrt{(\eta^2\beta^2 - \xi^2\gamma^2) k(2-k)}, \qquad \eta\beta > \xi\gamma, \\ &\rho\xi\gamma < 1 - k, \qquad k = \lambda + 2\sqrt{1 - 2\delta + \sigma^2}, \qquad k < 1. \end{aligned} \tag{5.4}$$

If $\lim_{n\to\infty} \|e_n\| = 0$, $\lim_{n\to\infty} \|f_n\| = 0$, then we have the following.

(I) *The sequence $\{u_n\}$ defined by Algorithm 5.1 converges strongly to the unique solution u^* of problem (2.5).*
(II) *Moreover, if $0 < a \leq \alpha_n$, then $\lim y_n = u^*$ if and only if $\lim \epsilon_n = 0$, where $\{\epsilon_n\}$ is defined by (5.2).*

PROOF. It follows from Theorem 4.1 that there exists $u^* \in H$ which is a solution of problem (2.5) and so

$$p(u^*) = J_\rho^{M(\cdot, u^*)} \left(p(u^*) - \rho N(Su^*, Tu^*) \right). \tag{5.5}$$

From (5.2), (5.5), and Algorithm 5.1, it follows that

$$\begin{aligned} \|u_{n+1} - u^*\| &= \Big\| (1-\alpha_n) u_n + \alpha_n \left[v_n - p(v_n) + J_\rho^{M(\cdot, v_n)} \left(p(v_n) - \rho N(Sv_n, Tv_n) \right) \right] + \alpha_n e_n \\ &\quad - (1-\alpha_n) u^* - \alpha_n \left[u^* - p(u^*) + J_\rho^{M(\cdot, u^*)} \left(p(u^*) - \rho N(Su^*, Tu^*) \right) \right] \Big\| \\ &\leq (1-\alpha_n) \|u_n - u^*\| + \alpha_n \|v_n - p(v_n) - (u^* - p(u^*))\| + \alpha_n \|e_n\| \\ &\quad + \alpha_n \left\| J_\rho^{M(\cdot, v_n)} \left(p(v_n) - \rho N(Sv_n, Tv_n) \right) - J_\rho^{M(\cdot, u^*)} \left(p(u^*) - \rho N(Su^*, Tu^*) \right) \right\| \\ &\leq (1-\alpha_n) \|u_n - u^*\| + \alpha_n \|v_n - u^* - (p(v_n) - p(u^*))\| + \alpha_n \|e_n\| \\ &\quad + \alpha_n \left\| J_\rho^{M(\cdot, v_n)} \left(p(v_n) - \rho N(Sv_n, Tv_n) \right) - J_\rho^{M(\cdot, v_n)} \left(p(u^*) - \rho N(Su^*, Tu^*) \right) \right\| \\ &\quad + \alpha_n \left\| J_\rho^{M(\cdot, v_n)} \left(p(u^*) - \rho N(Su^*, Tu^*) \right) - J_\rho^{M(\cdot, u^*)} \left(p(u^*) - \rho N(Su^*, Tu^*) \right) \right\| \\ &\leq (1-\alpha_n) \|u_n - u^*\| + 2\alpha_n \|v_n - u^* - (p(v_n) - p(u^*))\| + \alpha_n \|e_n\| \\ &\quad + \alpha_n \|v_n - u^* - \rho(N(Sv_n, Tu^*) - N(Su^*, Tu^*))\| + \lambda\alpha_n \|v_n - u^*\| \end{aligned} \tag{5.6}$$

$$\begin{aligned}&\leq (1-\alpha_n)\|u_n-u^*\| + 2\alpha_n \|v_n - u^* - (p(v_n) - p(u^*))\| + \alpha_n \|e_n\| \\ &\quad + \alpha_n \|v_n - u^* - \rho(N(Sv_n, Tv_n) - N(Su^*, Tv_n))\|" \qquad (5.6)(\text{cont.}) \\ &\quad + \alpha_n \rho \|N(Su^*, Tv_n) - N(Su^*, Tu^*)\| + \lambda\alpha_n \|v_n - u^*\|.\end{aligned}$$

By the Lipschitz continuity of N, S, T, p and the strong monotonicity of S and p, we obtain

$$\|v_n - u^* - (p(v_n) - p(u^*))\|^2 \leq (1-2\delta+\sigma^2)\|v_n - u^*\|^2, \tag{5.7}$$

$$\|v_n - u^* - \rho(N(Sv_n, Tv_n) - N(Su^*, Tv_n))\|^2 \leq (1-2\rho\alpha+\rho^2\eta^2\beta^2)\|v_n - u^*\|^2, \tag{5.8}$$

$$\|N(Su^*, Tv_n) - N(Su^*, Tu^*)\| \leq \xi\gamma \|v_n - u^*\|. \tag{5.9}$$

It follows from (5.6)–(5.9) that

$$\|u_{n+1} - u^*\| \leq (1-\alpha_n)\|u_n - u^*\| + \theta\alpha_n \|v_n - u^*\| + \alpha_n \|e_n\|, \tag{5.10}$$

where

$$\theta = \lambda + 2\sqrt{1-2\delta+\sigma^2} + \sqrt{1-2\rho\alpha+\rho^2\eta^2\beta^2} + \rho\xi\gamma.$$

Similarly, we have

$$\|v_n - u^*\| \leq (1-\beta_n)\|u_n - u^*\| + \theta\beta_n \|u_n - u^*\| + \beta_n \|f_n\|. \tag{5.11}$$

From (5.10) and (5.11), we have

$$\|u_{n+1} - u^*\| \leq [1 - \alpha_n(1-\theta)(1+\beta_n\theta)]\|u_n - u^*\| + \alpha_n\beta_n\theta \|f_n\| + \alpha_n \|e_n\|.$$

Condition (5.4) implies that $0 < \theta < 1$, and so

$$\|u_{n+1} - u^*\| \leq (1 - \alpha_n(1-\theta))\|u_n - u^*\| + \alpha_n(1-\theta)h_n, \tag{5.12}$$

where

$$h_n = \frac{\beta_n\theta \|f_n\| + \|e_n\|}{1-\theta} \to 0, \qquad \text{as } n \to \infty.$$

It follows from (5.12) and Lemma 5.1 that $u_n \to u^*$ as $n \to \infty$.

Now we prove that u^* is a unique solution of problem (2.5). In fact, if u is also a solution of problem (2.5), then

$$p(u) = J_\rho^{M(\cdot,u)}(p(u) - \rho N(Su, Tu))$$

and, as the proof of (5.10), we have

$$\|u^* - u\| \leq \theta \|u^* - u\|,$$

where $0 < \theta < 1$ and so $u^* = u$. This completes the proof of Conclusion I.

Next we prove Conclusion II. Using (5.2), we obtain

$$\begin{aligned}&\|y_{n+1} - u^*\| \\ &\leq \left\| y_{n+1} - \left\{(1-\alpha_n)y_n + \alpha_n\left[x_n - p(x_n) + J_\rho^{M(\cdot,x_n)}(p(x_n) - \rho N(Sx_n, Tx_n))\right] + \alpha_n e_n\right\}\right\| \\ &\quad + \left\|(1-\alpha_n)y_n + \alpha_n\left[x_n - p(x_n) + J_\rho^{M(\cdot,x_n)}(p(x_n) - \rho N(Sx_n, Tx_n))\right] + \alpha_n e_n - u^*\right\| \\ &= \left\|(1-\alpha_n)y_n + \alpha_n\left[x_n - p(x_n) + J_\rho^{M(\cdot,x_n)}(p(x_n) - \rho N(Sx_n, Tx_n))\right] + \alpha_n e_n - u^*\right\| + \epsilon_n.\end{aligned} \tag{5.13}$$

As the proof of inequality (5.12), we have

$$\begin{aligned}&\left\|(1-\alpha_n)y_n + \alpha_n\left[x_n - p(x_n) + J_\rho^{M(\cdot,x_n)}(p(x_n) - \rho N(Sx_n, Tx_n))\right] + \alpha_n e_n - u^*\right\| \\ &\qquad \leq (1 - \alpha_n(1-\theta))\|y_n - u^*\| + \alpha_n(1-\theta)h_n.\end{aligned} \tag{5.14}$$

Since $0 < a \le \alpha_n$, by (5.13) and (5.14), we have

$$\|y_{n+1} - u^*\| \le (1 - \alpha_n (1-\theta)) \|y_n - u^*\| + \alpha_n(1-\theta) \left[h_n + \frac{\epsilon_n}{a(1-\theta)}\right]. \tag{5.15}$$

Suppose that $\lim \epsilon_n = 0$. Then from $h_n \to 0$, $\sum \alpha_n = \infty$, (5.15), and Lemma 5.1, we have $\lim y_n = u^*$.

Conversely, suppose that $\lim y_n = u^*$. Then we have

$$\begin{aligned}
\epsilon_n &= \|y_{n+1} - \{(1-\alpha_n)y_n + \alpha_n[x_n - p(x_n) + J_\rho^{M(\cdot,x_n)}(p(x_n) - \rho N(Sx_n, Tx_n))] + \alpha_n e_n\}\| \\
&\le \|y_{n+1} - u^*\| + \Big\|(1-\alpha_n) y_n + \alpha_n \Big[x_n - p(x_n) + J_\rho^{M(\cdot,x_n)} (p(x_n) \\
&\quad -\rho N(Sx_n, Tx_n))\Big] + \alpha_n e_n - u^*\Big\| \\
&\le \|y_{n+1} - u^*\| + (1 - \alpha_n (1-\theta)) \|y_n - u^*\| + \alpha_n (1-\theta) h_n \to 0,
\end{aligned}$$

as $n \to \infty$. This completes the proof.

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